$SL_3(\mathbb{C}[X])$ DOES NOT HAVE BOUNDED WORD LENGTH

Wilberd van der Kallen

Introduction.

When R is a ring, we say that the elementary group $E_n(R)$ has bounded word length (with respect to elementary matrices) if there is an integer $\nu_n(R)$ such that each element of $E_n(R)$ can be written as a product of length at most $\nu_n(R)$, the factors in the product being elementary matrices. D. Carter and G. Keller have recently shown ([2]) that $SL_n(R)$ has bounded word length if R is the ring of integers in an algebraic number field and $n \geq 3$. (In this case $SL_n(R)$ equals $E_n(R)$.) As the K_2 of such a ring of integers is finite, their result implies that for $n \geq 4$ the Steinberg group $St_n(R)$ has bounded word length with respect to its usual generators $x_{ij}(r)$.

In this note we show that there is no bounded word length for $SL_n(k[X])$ if k is a field of infinite transcendence degree over its prime field and n is at least 2. We also draw attention to the question of bounded word length for $St_{n+4}(\mathbb{Z}[X_1,\ldots,X_n])$, which is still open for $n \geq 1$.

(1.1) Let R be a ring which is associative with 1.

Lemma (R. K. Dennis)

If $E_n(R)$ has bounded word length, $n \geq 2$, then $E_{n+1}(R)$ also has bounded word length. (Similar result for Steinberg groups.)

Sketch of proof. Instead of elementary matrices one may use unipotent triangular matrices (upper or lower triangular). Given that every element in $E_n(R)$ can be written as a product of N unipotent triangular matrices in $E_n(R)$, one shows that the set $\{g \in E_{n+1}(R)$: g can be written as a product of N unipotent triangular matrices in $E_{n+1}(R)\}$ is invariant under left multiplication by generators $e_{ij}(r)$ of $E_{n+1}(R)$ with $|i-j| = 1$.

Remark. A unipotent triangular matrix in $E_n(R)$ can be written as the product of three commutators. $(n \geq 3)$. A similar statement holds in $St_n(R)$.

(1.2) Following a suggestion from a logician, let us look at the canonical isomorphism $GL_n(R^{\mathbb{N}}) \longrightarrow GL_n(R)^{\mathbb{N}}$, where $X^{\mathbb{N}}$ denotes the infinite product $\prod_{i=1}^{\infty} X$ of copies of X. This isomorphism induces a map $K_1(n,R^{\mathbb{N}}) \longrightarrow K_1(n,R)^{\mathbb{N}}$ and it is easy to see that this map is injective if and only if $E_n(R)$ has bounded word length. Now suppose that s.r. $R < \infty$, i.e., that R satisfies a stable range condition. Then $R^{\mathbb{N}}$ satisfies the same stable range condition and we have $K_1(n,R^{\mathbb{N}}) \cong K_1(R^{\mathbb{N}})$, $K_1(n,R) \cong K_1(R)$ for $n \geq$ s.r. $R + 1$. It follows that if $E_n(R)$ has bounded word length for some n $(n \geq 2)$, it has bounded word length for $n \geq$ s.r. $R + 1$.

Note that $E(R) = E_\infty(R)$ never has bounded word length: There is no shorter way to write $e_{1,2}(1)e_{3,4}(1)\cdots e_{n,n+1}(1)$. If one considers word length with respect to commutators then one does get a bound for $E_\infty(R)$: Every element can be written as a product of four unipotent triangular matrices, hence of twelve commutators. (This also holds in $St_\infty(R)$.) Thus the question of bounded word length is more interesting for $E_n(R)$ (or $St_n(R)$) with n finite.

(1.3) Lemma. Let F be a field.

(i) If $St_n(F)$ has bounded word length, $n \geq 2$, then $K_2(F)$ has bounded word length in terms of the Steinberg symbols $\{u,v\}$.

(ii) Let $B \geq 1$ be an integer and assume that every element of $K_2(F)$ can be written as a product of B Steinberg symbols. Then the Milnor K-group $K_n^M(F)$ is annihilated by $2((B+1)!)$ for $n \geq 2B + 2$.

Proof. Part (i) follows from the Bruhat decomposition in $St_n(F)$. (cf. [5] Lemma 9.15)

Part (ii). We may assume $n = 2B + 2$. Let $\alpha = \ell(x_1) \cdot \cdot \cdot \ell(x_n) \in K_n^M(F)$. Rewrite the element $\beta = \ell(x_1)\ell(x_2) + \cdot \cdot \cdot + \ell(x_{n-1})\ell(x_n)$ in $K_2(F)$ $(\equiv K_2^M(F))$ as $\ell(y_1)\ell(y_2) + \cdot \cdot \cdot + \ell(y_{2B-1})\ell(y_{2B})$. Using that $2\ell(z)^2 = 0$ for all $z \in F^*$, we find that $2((B+1)!)\alpha = 2\beta^{B+1} = 0$.

(1.4) Remarks.

(1) In the proof of part (i) it is essential that F is something like a field, as one sees from the following example. Let k denote the algebraic closure of $\mathbb{Q}$ and put $F = k(X) \otimes_k k(Y)$. We view F as a localization of $k(X)[Y]$. The ring F is a 1-dimensional domain and it follows from a localization sequence argument that $K_2(F)$ is generated by Steinberg symbols. Using tame symbols one shows that the element $\alpha = \prod_{j=1}^{2n} \{X-j, j-Y\}$ of $K_2(F)$ cannot be written as a product of fewer than n Steinberg symbols in $K_2(F)$. However, it can be written as the single Steinberg symbol $\left\{ \prod_{j=1}^{2n} (\frac{j-X}{j-Y}), X-Y \right\}$ in the K_2 of the field of fractions $k(X,Y)$ of F. What is more, it can be written as a single Dennis-Stein symbol $\langle \pm(1 - \prod_{i=1}^{2n} (\frac{i-X}{i-Y}))/(X-Y), X-Y \rangle$ in $St_4(F)$. (The sign depends on a choice of conventions.) Thus α is an element with word length at least n in terms of Steinberg symbols, but with word length at most 6 in terms of the usual generators of $St_4(F)$.

(2) It follows from a theorem of H. W. Lenstra Jr. ([4]) that one may take $B = 1$ in part (ii) when F is a global field. (In fact the higher Milnor K-groups are known in this case ([1]) and they are annihilated by 2.) Recall also that it is tempting to conjecture that, if F is a field of Kronecker dimension $\delta(F)$ (i.e., if F has transcendence degree $\delta(F)-1$ over a global field), the Milnor K-group $K_n^M(F)$ is torsion for $n > \delta(F)$. (cf. [1] (5.10)).

(1.5) Proposition. Let k be a field such that $SL_n(k[X])$ $(= E_n(k[X]))$ has bounded word length for some $n \geq 2$. Then k has finite transcendence degree over its prime field.

<u>Proof</u>. By (1.1) we may assume $n \geq 3$. Say every element of $E_n(k[X])$ is the product of B elementary matrices. Consider the familiar exact sequence

$$K_2(k[X]) \to K_2(k[X]/(X^2-X)) \to K_1(k[X],(X^2-X)) \to K_1(k[X]).$$

The cokernel of the first map is $K_2(k)$ and that is therefore also the kernel of the last map. Tracing the proof of exactness of the sequence (cf. [5] Theorem 6.2) one sees that any element α of $K_2(k)$ can be represented, as an element of the cokernel of the first map, by an expression of length at most B in $St_n(k[X]/(X^2-X))$. Projecting down to $St_n(k)$ via $X \mapsto 0$, $X \mapsto 1$ respectively, and dividing the two results, we see that α can also be represented by an expression of length at most $2B$ in $St_n(k)$. Arguing as in (1.3) we conclude that $K_m^M(k)$ is a torsion group for m large. By ([6] Proposition 2) the result follows from this.

(2.1) If A, B are rings, then we say that A covers B if for every finite subset V of B there is a homomorphism $\phi: A \longrightarrow B$ with $V \subset \phi(A)$. Clearly, if A covers B and $E_n(A)$ has bounded word length, then $E_n(B)$ has bounded word length too. If R is commutative and S is a multiplicative subset, then the polynomial ring $R[X]$ covers $S^{-1}R$ because any finite subset of $S^{-1}R$ admits a common denominator. If F is a field of transcendence degree d over its prime field, then every finitely generated subfield of F is a monogenic (separable) extension of a purely transcendental extension of the prime field, hence $\mathbb{Z}[X_1,\ldots,X_{d+2}]$ covers F. Thus we are led to ask:

(Q_n): Does $E_{n+3}(\mathbb{Z}[X_1,\ldots,X_n])$ have bounded word length?

An equivalent question is:

(Q_n'): Does $St_{n+4}(\mathbb{Z}[X_1,\ldots,X_n])$ have bounded word length?

(2.2) Note that for symplectic groups the answer to the analogue of the question (Q_0') is known to be negative: Let τ be the continuous symplectic symbol $K_2^{\text{sympl.}}(\mathbb{R}) \to \mathbb{Z}$. The surjective map $K_2^{\text{sympl.}}(\mathbb{Z}) \to K_2^{\text{sympl.}}(\mathbb{R}) \overset{\tau}{\to} \mathbb{Z}$ sends expressions of bounded length via products of bounded length of symplectic Steinberg symbols to a bounded subset of $\mathbb{Z}$.

In particular this shows that there is no bounded word length in $St_2(\mathbb{Z})$, but that is clear anyway, because it is a classical result, related to the theory of continued fractions, that even $SL_2(\mathbb{Z})$ does not have bounded word length. (Compare also [3] §8.)

References.

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Mathematisch Instituut
der Rijksuniversiteit te Utrecht
Budapestlaan, De Uithof
Utrecht, The Netherlands