Cohomology with Grosshans graded coefficients

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Abstract

Let the reductive group G act on the finitely generated commutative k-algebra A. We ask if the finite generation property of the ring of invariants A^G extends to the full cohomology ring $H^*(G, A)$. We confirm this for $G = SL_2$ and also when the action on A is replaced by the 'contracted' action on the Grosshans graded ring $\operatorname{gr} A$, provided the characteristic of k is large.

1 Introduction

Consider a linear algebraic group or group scheme G defined over an algebraically closed field k of positive characteristic p. Let A be a finitely generated commutative k-algebra on which G acts rationally by k-algebra automorphisms. So G acts on Spec(A). One may then ask if the cohomology ring $H^*(G, A)$ is finitely generated as a k-algebra.

We think of this as a question in invariant theory. Our question is not what combinations of G and A yield a finitely generated $H^*(G, A)$. Instead we are interested in finding those G for which every A as above will give a finitely generated $H^*(G, A)$. In particular, G should be such that for all these A the ring of invariants $A^G = H^0(G, A)$ is finitely generated. This is why we must restrict attention to geometrically reductive G. We believe no further restriction is needed, and our aim is to present some evidence for this. In characteristic zero there would be nothing to do. Indeed, suppose G is a reductive linear algebraic group over $\mathbb{C}$. We are concerned with representations of the algebraic group, the so-called rational representations. The group is linearly reductive, meaning that all extensions of representations split (even when the representations are infinite dimensional). So there is no higher rational cohomology and $H^*(G, A) = A^G$. Thus our problem asks nothing new in this case. That is why we now return to characteristic p. Notice that in characteristic p there is one type of module or algebra for which the invariants, including the higher invariants known as cohomology, behave as in characteristic zero. They are the modules/algebras with good filtration. They will serve as a natural tool in the sequel.

Our proofs combine arguments and results from several earlier works. As our question concerns 'higher invariant theory', it is clear that invariant theory will play its part. On the other hand our work is a direct descendent of the work of Evens for the case that G is a finite group, and the work of Friedlander and Suslin for the case of finite group schemes. We try to merge this strand with invariant theory and emphasize that in both cases G happens to be geometrically reductive.

Thus say G is geometrically reductive. Then we know by Nagata that at least the ring of invariants $A^G = H^0(G, A)$ is finitely generated. (As explained in [4], Nagata's proof [21] extends to group schemes.)

By Waterhouse [28] a finite group scheme is geometrically reductive. If G is a finite group scheme, then A is a finite module over A^G and hence the cohomology ring $H^*(G, A)$ is indeed finitely generated by Friedlander and Suslin [11]. (If G is finite reduced, see Evens [8, Thm. 8.1]. If G is finite and connected, take $C = A^G$ in [11, Theorem 1.5, 1.5.1]. If G is not connected, one finishes the argument by following [8] as on pages 220–221 of [11].)

Note that if the geometrically reductive G is a subgroup of GL_n, then GL_n/G is affine, $\mathrm{ind}_G^{GL_n} A$ is finitely generated, and $H^*(G, A) = H^*(GL_n, \mathrm{ind}_G^{GL_n} A)$. (Compare [21], [23], [13, Ch. II], [15, I 4.6, I 5.13].) Therefore let us now assume $G = GL_n$, or rather $G = SL_n$ to keep it semi-simple. (The SL_n case suffices, as $H^*(GL_n, A) = H^*(SL_n, A)^{\mathbb{G}_m}$ for a GL_n-algebra A.)

Grosshans [12] has introduced a filtration on A. The associated graded ring $\mathrm{gr}\,A$ is finitely generated [12, Lemma 14]. There is a flat family with general fibre A and special fibre $\mathrm{gr}\,A$ [12, Theorem 13]. Its counterpart in characteristic zero was introduced by Popov [22]. Our first main result says

Theorem 1.1 *If $n < 6$ or $p > 2^n$, then $H^*(SL_n, \mathrm{gr}\,A)$ is finitely generated as a k-algebra.*

Remark 1.2 In problems 3.10 and 3.11 we discuss how one could try to remove the gr in the conclusion of the theorem.

Our second main result concerns the case $n = 2$, where we succeed in removing the gr, using a family of universal cohomology classes which behaves as if it is obtained by taking divided powers of the class e_1 of Friedlander and Suslin [11].

Theorem 1.3 (Cohomological invariant theory in rank one) *Let A be a finitely generated commutative k-algebra on which SL_2 acts rationally by algebra automorphisms. Then $H^*(SL_2, A)$ is finitely generated as a k-algebra.*

Remark 1.4 We know very little about the size of $H^*(G, A)$, even in simple examples. For instance, let $p = 2$ and consider the second Steinberg module $V = \mathrm{St}_2$ of SL_2. The dimension of V is four. Its highest weight is three times the fundamental weight. By the theorem $H^*(SL_2, S^*(V))$ is finitely generated, where $S^*(V)$ denotes the symmetric algebra. But that is all we know about the size of $H^*(SL_2, S^*(V))$. Note that $S^*(V)$ does not have a good filtration [5, p.71 Example], as $H^1(SL_2, S^2(V)) \neq 0$.

2 Recollections

For simplicity we stay with the important case $G = SL_n$ until 3.13. We choose a Borel group $B^+ = TU^+$ of upper triangular matrices and the opposite Borel group B^-. The roots of B^+ are positive. If $\lambda \in X(T)$ is dominant, then $\mathrm{ind}_{B^-}^G(\lambda)$ is the dual Weyl module $\nabla_G(\lambda)$ with highest weight λ. In a good filtration of a G-module the layers are of the form $\nabla_G(\mu)$. As in [27] we will actually also allow a layer to be a direct sum of any number of copies of the same $\nabla_G(\mu)$. If M is a G-module, and $m \geq -1$ is an integer so that $H^{m+1}(G, \nabla_G(\mu) \otimes M) = 0$ for all dominant μ, then we say as in [10] that M has *good filtration dimension* at most m. The case $m = 0$ corresponds with M having a good filtration. We say that M has good filtration dimension precisely m, notation $\dim_\nabla(M) = m$, if m is minimal so that M has good filtration dimension at most m. In that case $H^{i+1}(G, \nabla_G(\mu) \otimes M) = 0$ for all dominant μ and all $i \geq m$. In particular $H^{i+1}(G, M) = 0$ for $i \geq m$. If there is no finite m so that $\dim_\nabla(M) = m$, then we put $\dim_\nabla(M) = \infty$.

Lemma 2.1 *Let $0 \to M' \to M \to M'' \to 0$ be exact. Then*

1. $\dim_\nabla(M) \leq \max(\dim_\nabla(M'), \dim_\nabla(M''))$,
2. $\dim_\nabla(M') \leq \max(\dim_\nabla(M), \dim_\nabla(M'') + 1)$,
3. $\dim_\nabla(M'') \leq \max(\dim_\nabla(M), \dim_\nabla(M') - 1)$,
4. $\dim_\nabla(M' \otimes M'') \leq \dim_\nabla(M') + \dim_\nabla(M'')$.

2.2 Filtrations

If M is a G-module, and λ is a dominant weight, then $M_{\leq\lambda}$ denotes the largest G-submodule all whose weights μ satisfy $\mu \leq \lambda$ in the usual partial order [15, II 1.5]. Similarly $M_{<\lambda}$ denotes the largest G-submodule all whose weights μ satisfy $\mu < \lambda$. As in [27], we form the $X(T)$-graded module

$$\operatorname{gr}_{X(T)} M = \bigoplus_{\lambda \in X(T)} M_{\leq\lambda}/M_{<\lambda}.$$

Each $M_{\leq\lambda}/M_{<\lambda}$, or $M_{\leq\lambda/<\lambda}$ for short, has a B^+-socle $(M_{\leq\lambda/<\lambda})^U = M^U_\lambda$ of weight λ. We view M^U as a B^--module through restriction (inflation) along the homomorphism $B^- \to T$. Then $M_{\leq\lambda/<\lambda}$ embeds naturally in its 'good filtration hull' $\mathrm{hull}_\nabla(M_{\leq\lambda/<\lambda}) = \operatorname{ind}_{B^-}^G M^U_\lambda$. This good filtration hull has the same B^+-socle and is the injective hull in the category C_λ of G-modules N that satisfy $N = N_{\leq\lambda}$. Compare [27, 3.1.10].

Let us apply this in particular to our finitely generated commutative k-algebra with G action A. We get an $X(T)$-graded algebra $\operatorname{gr}_{X(T)} A$. We convert it to a $\mathbb{Z}$-graded algebra through an additive height function $\mathrm{ht} : X(T) \to \mathbb{Z}$, defined by $\mathrm{ht}(\gamma) = 2\sum_{\alpha>0}\langle\gamma, \widehat{\alpha}\rangle$, the sum being over the positive roots. (Our ht is twice the one used by Grosshans, because we prefer to get even degrees rather than just integer degrees.) The Grosshans graded algebra is now

$$\operatorname{gr} A = \bigoplus_{i \geq 0} \operatorname{gr}_i A,$$

with

$$\operatorname{gr}_i A = \bigoplus_{\mathrm{ht}(\lambda)=i} A_{\leq\lambda/<\lambda}.$$

It embeds in a good filtration hull, which Grosshans calls R, and which we call $\mathrm{hull}_\nabla(\operatorname{gr} A)$,

$$\mathrm{hull}_\nabla(\operatorname{gr} A) = \operatorname{ind}_{B^-}^G A^U = \bigoplus_i \bigoplus_{\mathrm{ht}(\lambda)=i} \mathrm{hull}_\nabla(A_{\leq\lambda}/A_{<\lambda}).$$

Grosshans shows that A^U, $\operatorname{gr} A$, $\mathrm{hull}_\nabla(\operatorname{gr} A)$ are finitely generated with $\mathrm{hull}_\nabla(\operatorname{gr} A)$ finite over $\operatorname{gr} A$. Mathieu did a little better in [19]. His argument shows that in fact $\mathrm{hull}_\nabla(\operatorname{gr} A)$ is a p-root closure of $\operatorname{gr} A$. That is,

Lemma 2.3 *For every $x \in \mathrm{hull}_\nabla(\operatorname{gr} A)$, there is an integer $r \geq 0$, so that $x^{p^r} \in \operatorname{gr} A$.*

Proof It suffices to take $x \in \mathrm{hull}_\nabla(A_{\leq\lambda}/A_{<\lambda})$ for some λ. If $\lambda = 0$, then $\mathrm{hull}_\nabla(A_{\leq\lambda}/A_{<\lambda}) = A^G = \mathrm{gr}_0 A$. So say $\lambda > 0$ and consider the subalgebra $S = k \oplus \bigoplus_{i>0} \mathrm{hull}_\nabla(A_{\leq i\lambda}/A_{<i\lambda})$ of $\mathrm{hull}_\nabla(\mathrm{gr}\, A)$, with its subalgebra $S \cap \mathrm{gr}\, A$. Apply [27, Thm 4.2.3] to conclude that the p-root closure of $S \cap \mathrm{gr}\, A$ in S has a good filtration. As it contains all of S^U, it must be S itself. $\square$

Example 2.4 Consider the multicone [16]

$$k[G/U] = \mathrm{ind}_U^G k = \mathrm{ind}_{B^+}^G \mathrm{ind}_U^{B^+} k = \mathrm{ind}_{B^+}^G k[T] = \bigoplus_{\lambda \text{ dominant}} \nabla_G(\lambda).$$

It is its own Grosshans graded ring. Recall [16] that it is finitely generated by the sum of the $\nabla(\varpi_i)$, where ϖ_i denotes the ith fundamental weight. Together with the transfer principle $A^U \cong (k[G/U] \otimes A)^G$, see [13, Ch Two], this gives finite generation of A^U.

3 Proof of Theorem 1.1

Choose r so big that $x^{p^r} \in \mathrm{gr}\, A$ for all $x \in \mathrm{hull}_\nabla(\mathrm{gr}\, A)$. We may view $\mathrm{gr}\, A$ as a finite $\mathrm{hull}_\nabla(\mathrm{gr}\, A)^{(r)}$-module, where the exponent (r) denotes an rth Frobenius twist [15, I 9.2].

3.1 Key hypothesis

We assume that for every fundamental weight ϖ_i the symmetric algebra $S^*(\nabla(\varpi_i))$ has a good filtration.

The hypothesis in theorem 1.1 is explained by

Lemma 3.2 *If $n < 6$ or $p > 2^n$, then the key hypothesis is satisfied.*

Proof We follow [1, section 4]. If $p > 2^n$, then $p > \sum_i \dim(\nabla(\varpi_i))$, so $\wedge^j(\nabla(\varpi_i))$ has a good filtration for all j by [1], so $S^*(\nabla(\varpi_i))$ has a good filtration by [1]. If $n < 6$, then by symmetry of the Dynkin diagram (contragredient duality) it suffices to consider ϖ_1 and ϖ_2. But $S^*(\nabla(\varpi_1))$ has a good filtration for every n because $\wedge^j(\nabla(\varpi_1))$ has a good filtration for all j. And $S^*(\nabla(\varpi_2))$ has a good filtration for every n by Boffi [3]. (Thanks to J. Weyman and T. Jozefiak for pointing this out.) $\square$

Example 3.3 (J. Weyman) If $p = 2$ and $n \geq 6$, there is a submodule with highest weight ϖ_6 in $S^2(\nabla(\varpi_3))$. (If $n = 6$, read zero for ϖ_6, as we are working with SL_6 rather than GL_6.) With a character computation this implies that $S^2(\nabla(\varpi_3))$ does not have a good filtration. The submodule in question is generated by a highest weight vector which is a sum of ten terms $(e_1 \wedge e_{\sigma(2)} \wedge e_{\sigma(3)})(e_{\sigma(4)} \wedge e_{\sigma(5)} \wedge e_{\sigma(6)})$. One sums over the ten permutations σ of $2, 3, 4, 5, 6$ that satisfy $\sigma(2) < \sigma(3)$ and $\sigma(4) < \sigma(5) < \sigma(6)$.

We want to view $S^*(\nabla(\varpi_i))$ as a *graded polynomial G-algebra with good filtration*. Let us collect the properties that we will use in a rather artificial definition.

Definition 3.4 Let D be a diagonalizable group scheme [15, I 2.5]. We say that P is a graded polynomial $G \times D$-algebra with good filtration, if the following holds. First of all P is a polynomial algebra over k in finitely many variables. Secondly, these variables are homogeneous of non-negative integer degree, thus making P into a graded k-algebra. There is also given an action of $G \times D$ on P by algebra automorphisms that are compatible with the grading, with $G \times D$ acting trivially on the degree zero part P_0 of P. This P_0 is thus the polynomial algebra generated by the variables of degree zero. The variables of positive degree generate their own polynomial algebra P^c, which we also assume to be $G \times D$ invariant. Thus $P = P_0 \otimes_k P^c$. And finally, P^c is an algebra with a good filtration for the action of G. Then of course, so is P.

Example 3.5 Our key hypothesis makes that one gets a *graded polynomial $G \times D$-algebra with good filtration* by taking for D any subgroup scheme of T, with T acting on $P = S^*(\nabla(\varpi_i))$ through its natural $X(T)$-grading: On $S^j(\nabla(\varpi_i))$ we make T act with weight $j\varpi_i$.

If P_i are graded polynomial $G \times D_i$-algebras with good filtration for $i = 1, 2$, then $P_1 \otimes P_2$ is a graded polynomial $G \times (D_1 \times D_2)$-algebra with good filtration.

Definition 3.6 If P is a graded polynomial $G \times D$-algebra with good filtration, then by a *finite graded P-module M* we mean a finitely generated $\mathbb{Q}$-graded module for the graded polynomial ring, together with a $G \times D$ action on M which is compatible with the grading and with the action on P. It is not required that the action is trivial on the degree zero part of M. We call M *free* if it is free as a module over the polynomial ring. We call M *extended* if there is a finitely generated $\mathbb{Q}$-graded P_0-module V with $G \times D$ action, so that $M = V \otimes_{P_0} P$ as graded P-modules.

Lemma 3.7 *Let P be a graded polynomial $G \times D$-algebra with good filtration, and let M be a finite graded P-module.*

1. *There is a finite free resolution*
$$0 \to F_s \to F_{s-1} \to \cdots \to F_0 \to M.$$
2. *Every finite free P-module has a finite filtration whose quotients are extended.*
3. $\dim_\nabla(M) < \infty$.
4. *$H^i(G, M)$ is a finite P^G-module for every $i \geq 0$.*

Proof Take a finite dimensional graded $G \times D$-submodule V of M that generates M as a P-module. Then $F_0 = V \otimes_k P$ is free and it maps onto M. As M has finite projective dimension [20, 18C], we may repeat until the syzygy is projective. But a projective module over a polynomial ring is free by Quillen and Suslin [17].

Now consider a free P-module F. Let P_+ be the ideal generated by the P_i with $i > 0$. If F is free of rank t, then F/P_+F is free of rank t over P_0. Choose homogeneous elements $e_1, \ldots, e_t$ in F so that their classes form a basis of F/P_+F. Then the e_i generate F (cf. [14, 6.13 Lemma 5]), so they also form a basis of F. Let F_m be the component of lowest degree in F, assuming $F \neq 0$. The e_i that lie in F_m form a basis of F_m over P_0. So F_m generates a free, extended submodule PF_m of F and the quotient F/PF_m is free of lower rank.

To show that $\dim_\nabla(M) < \infty$, it suffices to consider the case of an extended module $M = V \otimes_{P_0} P$. As $V \otimes_{P_0} P = V \otimes_k P^c$, it suffices to check that V has finite good filtration dimension. But V is a finitely generated P_0-module and thus has only finitely many weights. Therefore the argument used in [10] to show that finite dimensional G modules have finite good filtration dimension, applies to V.

Finally let M be any finite graded P-module again. As G is reductive by Haboush [15, 10.7], it is well known that $H^0(G, M)$ is a finite P^G-module, because $(S^*_P(M))^G$ is finitely generated. So we argue by dimension shift. We claim that for s sufficiently large the module $M \otimes \mathrm{St}_s \otimes \mathrm{St}_s$ has a good filtration. It suffices to check this for the extended case $M = V \otimes_{P_0} P$ and then one can use again that V has only finitely many weights, so that one may choose s so large that all weights of $V \otimes k_{-(p^s-1)\rho}$ are anti-dominant. Then $V \otimes \mathrm{St}_s = \mathrm{ind}_{B^+}^G(V \otimes k_{-(p^s-1)\rho})$ has a good filtration and so does $M \otimes \mathrm{St}_s \otimes \mathrm{St}_s$. Then $H^i(G, M)$ is the cokernel of $H^{i-1}(G, M \otimes \mathrm{St}_s \otimes \mathrm{St}_s) \to H^{i-1}(G, M \otimes \mathrm{St}_s \otimes \mathrm{St}_s /M)$ for $i \geq 1$. □

Recall that we choose r so big that $x^{p^r} \in \operatorname{gr} A$ for all $x \in \operatorname{hull}_\nabla(\operatorname{gr} A)$. Let G_r denote the r-th Frobenius kernel. We will need multiplicative structure on a Hochschild–Serre spectral sequence. See for instance [2], translating from modules over a Hopf algebra to comodules over a Hopf algebra.

Proposition 3.8 *Assume the key hypothesis 3.1. With r as indicated, the spectral sequence*

$$E_2^{ij} = H^i(G/G_r, H^j(G_r, \operatorname{gr} A)) \Rightarrow H^{i+j}(G, \operatorname{gr} A)$$

stops, i.e. $E_s = E_\infty$ *for some* $s < \infty$.
In fact, $H^*(G_r, \operatorname{gr} A)^{(-r)}$ *has finite good filtration dimension.*
Moreover, E_2^{**} *is a finite module over the even part of* E_2^{0*}.

Proof By [11, Th 1.5, 1.5.1] the ring $H^*(G_r, \operatorname{gr} A)^{(-r)}$ is a finite module over the algebra

$$\bigotimes_{a=1}^{r} S^*((\mathfrak{gl}_n)^\#(2p^{a-1})) \otimes \operatorname{hull}_\nabla(\operatorname{gr} A).$$

Here $(\mathfrak{gl}_n)^\#(2p^{a-1})$ denotes the dual of $\mathfrak{gl}_n$ placed in degree $2p^{a-1}$. As $G = SL_n$, it follows from [1, 4.3] that the algebra $\bigotimes_{a=1}^{r} S^*((\mathfrak{gl}_n)^\#(2p^{a-1}))$ is a graded polynomial $G\times\{1\}$-algebra with good filtration. So we now try to replace $\operatorname{hull}_\nabla(\operatorname{gr} A)$ by a similarly nice algebra. That is, we seek a graded polynomial $G\times D$-algebra P with good filtration and a surjection $P^D \to \operatorname{hull}_\nabla(\operatorname{gr} A)$ of graded G-algebras. This is where the key hypothesis comes in. Let v be a weight vector in $\operatorname{hull}_\nabla(\operatorname{gr} A)^U = A^U$. If v has weight zero, we may map x to v, where x is the variable in a polynomial ring $k[x]$ with trivial grading and trivial $G \times T$ action. If v has weight $\lambda \neq 0$, take for D the scheme theoretic kernel of λ and observe that by our hypothesis the $X(T)$-graded algebra

$$P = \bigotimes_{i=1}^{n-1} S^*(\nabla(\varpi_i))$$

is a graded polynomial $G\times D$-algebra with good filtration, if we give $\nabla(\varpi_i)$ degree $\operatorname{ht}(\varpi_i)$. We have $P^D = \bigoplus_j P_{j\lambda}$, where $P_{j\lambda}$ denotes the summand $\bigotimes_{i=1}^{n-1} S^{m_i}(\nabla(\varpi_i))$ with $\sum_i m_i\varpi_i = j\lambda$. Choose a weight vector x of weight λ in P_λ (for the G action). The map from the polynomial ring $k[x]$ to $\operatorname{hull}_\nabla(\operatorname{gr} A)^U$ which sends x to v extends uniquely to a G equivariant algebra map $P^D \to \operatorname{hull}_\nabla(\operatorname{gr} A)$ because $P_{j\lambda} = (P_{j\lambda})_{\leq j\lambda}$. (The first subscript in the right hand side refers to a T action associated with the $X(T)$-grading, the second to the G action on P.) Compare [27, 4.2.4]. As A^U is finitely generated, we may combine finitely many such maps

$P(i)^{D(i)} \to \mathrm{hull}_\nabla(\mathrm{gr}\, A)$ into one surjective map $P^D \to \mathrm{hull}_\nabla(\mathrm{gr}\, A)$, with D the product of the $D(i)$ and P the tensor product of the $P(i)$. If we let the linearly reductive D act on $P \otimes_{P^D} H^*(G_r, \mathrm{gr}\, A)^{(-r)}$ through its action on P, then $H^*(G_r, \mathrm{gr}\, A)^{(-r)}$ is just the direct summand, as a G-module, consisting of the D-invariants. So in order to show that $H^*(G_r, \mathrm{gr}\, A)^{(-r)}$ has finite good filtration dimension, it suffices to show that $P \otimes_{P^D} H^*(G_r, \mathrm{gr}\, A)^{(-r)}$ has finite good filtration dimension. We now view $P \otimes \bigotimes_{a=1}^r S^*((\mathfrak{gl}_n)^\#(2p^{a-1}))$ as a graded polynomial $G \times D$-algebra with good filtration. (Collect the bigrading into a single total grading.) The algebra $P \otimes_{P^D} H^*(G_r, \mathrm{gr}\, A)^{(-r)}$ is a finite graded $P \otimes \bigotimes_{a=1}^r S^*((\mathfrak{gl}_n)^\#(2p^{a-1}))$-module. (We need a $\mathbb{Q}$-grading on $P \otimes_{P^D} H^*(G_r, \mathrm{gr}\, A)^{(-r)}$ because of the twist.) By Lemma 3.7 such a P-module has finite good filtration dimension. Therefore there are only finitely many i with $E_2^{i*} \neq 0$ and the spectral sequence stops. By the same lemma $H^i(G, P \otimes_{P^D} H^*(G_r, \mathrm{gr}\, A)^{(-r)})$ is finite over $H^0(G, P \otimes \bigotimes_{a=1}^r S^*((\mathfrak{gl}_n)^\#(2p^{a-1})))$. Taking D-invariants again, we see that E_2^{**} is a finite module over $H^0(G/G_r, (P^D \otimes \bigotimes_{a=1}^r S^*((\mathfrak{gl}_n)^\#(2p^{a-1})))^{(r)})$. But this ring acts by way of the the even part of E_2^{0*}. □

3.9 End of proof of theorem 1.1

Look some more at the spectral sequence of Proposition 3.8. We argue partly as in Evens' proof of his finite generation theorem [2, 4.2], [8]. As we have already observed, it follows from [11] that $H^*(G_r, \mathrm{gr}\, A)$ is noetherian over its finitely generated even degree part. By the proposition E_2 is noetherian over the even degree part of E_2^{0*}, which is finitely generated because G/G_r is reductive. Or, recall from the proof of the proposition that the E_2 term is finite over the finitely generated k-algebra $H^0(G/G_r, (P^D \otimes \bigotimes_{a=1}^r S^*((\mathfrak{gl}_n)^\#(2p^{a-1})))^{(r)})$. Now the E_2 term is a differential graded algebra in characteristic p, so the pth power of an element of the even part passes to the next term in the spectral sequence. Therefore E_3 is also noetherian over its finitely generated even degree part. As the spectral sequence stops, we get by repeating this argument that E_∞ is finitely generated. But then so is the abutment. □

Problem 3.10 There is a spectral sequence

$$E_1^{ij} = H^{i+j}(G, \mathrm{gr}_{-i}\, A) \Rightarrow H^{i+j}(G, A),$$

and the theorem says that E_1 is finitely generated. The generators are in bidegree $(0,0)$ or (i, j) with $i + j > 0$, $i < 0$. We would like this spectral sequence to stop too. It would suffice to know that if J is a G-stable ideal in a ring like A, then the even part of $H^*(G, A/J)$ is integral over the image of the even part of $H^*(G, A)$. Of course this integrality would be implied by finite generation of $H^*(G, S^*_A(A/J))$. But recall that Nagata proves first that $H^0(G, A/J)$ is integral over $H^0(G, A)$. So one could hope for a direct proof.

Problem 3.11 If M is a vector space, let $\Gamma^*(M) = \bigoplus_m (S^m(M^\#))^\#$ denote its divided power algebra. It is probably too much to ask for a divided power structure on the even part of the bigraded algebra $H^*(GL_n, \Gamma^*((\mathfrak{gl}_n)(2)^{(1)}))$, extending the divided power structure [7, Appendix A2], [24] on the graded algebra $H^0(GL_n, \Gamma^*((\mathfrak{gl}_n)(2)^{(1)}))$. But such a structure would be very helpful, as it would explain and enrich the supply of universal cohomology classes from [11]. And $H^{i+j}(G, \mathrm{gr}_{-i} A) \Rightarrow H^{i+j}(G, A)$ would then undoubtedly stop. See our treatment of the rank one case below.

Problem 3.12 To improve on the conditions of theorem 1.1, rather than on its conclusion, one should try and prove the following. If our finitely generated A has a good filtration and M is a finite A-module on which G acts compatibly, then $H^*(G, M)$ is finite over A^G. Of course this would again be implied by finite generation of $H^*(G, S^*_A(M))$.

Remark 3.13 Let G be a semi-simple group defined over k, and let V be a tilting G-module [6] of dimension n. Choose a basis in V and assume that the representation is faithful, so that G can be identified with a subgroup of SL_n. Assume $p > n/2$. Then $S^*((\mathfrak{gl}_n)^\#)$ has a good filtration as a G-module [1, 4.3]. Also assume for every fundamental weight ϖ_i of G that the symmetric algebra $S^*(\nabla(\varpi_i))$ has a good filtration. This happens for instance when p exceeds the dimensions of the fundamental representations of G. Let A be a finitely generated commutative k-algebra on which G acts rationally by k-algebra automorphisms. Then again $H^*(G, \mathrm{gr}\, A)$ is finitely generated as a k-algebra. The proof is the same as the proof of theorem 1.1.

4 Divided powers

Let $W_2(k) = W(k)/p^2W(k)$ be the ring of Witt vectors of length two over k, see [25, II §6]. One has an extension of algebraic groups

$$1 \to \mathfrak{gl}_n^{(1)} \to GL_n(W_2(k)) \to GL_n(k) \to 1,$$

whence a cocycle class $e_1 \in H^2(GL_n, \mathfrak{gl}_n^{(1)})$. We call it the Witt vector class for GL_n. Analogously there are Witt vector extensions and Witt vector classes for other groups that are originally defined over the integers, and for Frobenius kernels in them.

Remark 4.1 Observe that if $x_\alpha : \mathbb{G}_a \to SL_n$ is a root homomorphism corresponding to the root α, then the restriction of e_1 to $\mathbb{G}_a$ is non-trivial. In fact one may restrict further to the Frobenius kernel $\mathbb{G}_{a1}$. This $\mathbb{G}_{a1}$ acts trivially on $\mathfrak{gl}_n^{(1)}$, and the T-equivariant projection of $\mathfrak{gl}_n^{(1)}$ onto its $p\alpha$ weight space is thus also $\mathbb{G}_{a1}$-equivariant. Altogether we get an image β of e_1 in $H^2(\mathbb{G}_{a1}, (\mathfrak{gl}_n^{(1)})_{p\alpha})^T$. It is just the Witt vector class of $\mathbb{G}_{a1}$. It is well known that this class is non-trivial, compare [15, I 4.22, I 4.25], [11, §6]. Also note that the image of β in $H^2(\mathbb{G}_{a1}, \mathfrak{gl}_n^{(1)})$ under the map induced by the inclusion of $(\mathfrak{gl}_n^{(1)})_{p\alpha}$ into $\mathfrak{gl}_n^{(1)}$ is the same as the restriction of e_1 to $\mathbb{G}_{a1}$.

Lemma 4.2 *The Witt vector class for GL_n coincides with the universal cohomology class e_1 of Friedlander and Suslin, up to a non-zero scalar factor.*

Proof Using [11, Remark 1.2.1, Corollary 3.13] we see it suffices to take $r = j = 1$ and $q = 2$ in [11, Theorem 4.5]. □

4.3 Divided powers in rank one.

If R is a commutative k-algebra, and M is a finite dimensional vector space over k, then the divided power algebra $R \otimes \Gamma^*(M) = \Gamma^*_R(R \otimes M)$ is a ring with divided powers ([7, Appendix A2]). We write $\Delta_{i,j}$ for the component $\Gamma^{i+j}(M) \to \Gamma^i(M) \otimes \Gamma^j(M)$ of the comultiplication map $\Gamma^*(M) \to \Gamma^*(M) \otimes \Gamma^*(M)$. So $\Delta_{i,j}$ is the dual of the multiplication map $S^i(M^\#) \otimes S^j(M^\#) \to S^{i+j}(M^\#)$. If $v \in M$ has divided powers $v^{[i]} \in \Gamma^i(M)$, then $\Delta_{i,j}(v^{[i+j]}) = v^{[i]} \otimes v^{[j]}$. Put $G = GL_n$ and define T, B^+, B^- as usual. Actually n will be two, but we optimistically keep writing n. Let $r \geq 1$. As G_r acts trivially on $\mathfrak{gl}_n^{(r)}$, there is a divided power structure on $H^{\mathrm{even}}(G_r, \Gamma^*(\mathfrak{gl}_n^{(r)})) = H^{\mathrm{even}}(G_r, k) \otimes \Gamma^*(\mathfrak{gl}_n^{(r)})$, with the m-th divided power operation going from $H^{2a}(G_r, \Gamma^b(\mathfrak{gl}_n^{(r)}))$ to $H^{2am}(G_r, \Gamma^{bm}(\mathfrak{gl}_n^{(r)}))$ for $b \geq 1$.

It would be nice to extend the next theorem to arbitrary n. We do not know how to put a divided power structure on $\bigoplus_m H^{2m}(GL_n, \Gamma^m(\mathfrak{gl}_n^{(1)}))$. Nevertheless we feel the theorem is named appropriately.

Theorem 4.4 (Divided powers in rank one) *Let $n = 2$. There are classes $c[m] \in H^{2m}(GL_n, \Gamma^m(\mathfrak{gl}_n^{(1)}))$ so that*

1. *$c[1]$ is the Witt vector class e_1,*

2. *$(\Delta_{i,j})_*(c[i + j]) = c[i] \cup c[j]$ for $i, j \geq 1$.*

Proof Let α be the negative root, and let $x_\alpha : \mathbb{G}_a \to GL_n$ be its root homomorphism. By Kempf vanishing $H^{2m}(GL_n, \Gamma^m(\mathfrak{gl}_n^{(1)})) = H^{2m}(\mathbb{G}_a, \Gamma^m(\mathfrak{gl}_n^{(1)}))^T$, see [15, II 4.7c]. So we restrict to $\mathbb{G}_a$ along x_α. As $\mathbb{G}_a$ acts trivially on $\Gamma^*((\mathfrak{gl}_n^{(1)})_{p\alpha})$, we also have a divided power structure on $H^{\text{even}}(\mathbb{G}_a, \Gamma^*((\mathfrak{gl}_n^{(1)})_{p\alpha})) = H^{\text{even}}(\mathbb{G}_a, k) \otimes \Gamma^*((\mathfrak{gl}_n^{(1)})_{p\alpha})$. Take the m-th divided power in $H^{\text{even}}(\mathbb{G}_a, \Gamma^*((\mathfrak{gl}_n^{(1)})_{p\alpha}))^T$ of the Witt vector class of $\mathbb{G}_a$ and map it to $H^{2m}(\mathbb{G}_a, \Gamma^m(\mathfrak{gl}_n^{(1)}))^T$ using the inclusion of $(\mathfrak{gl}_n^{(1)})_{p\alpha}$ into $\mathfrak{gl}_n^{(1)}$. One lands in $H^{2m}(B^-, \Gamma^m(\mathfrak{gl}_n^{(1)})) \cong H^{2m}(G, \Gamma^m(\mathfrak{gl}_n^{(1)}))$. This gives the desired class $c[m]$. Indeed $(\Delta_{i,j})_*(c[i + j]) = c[i] \cup c[j]$ holds in the context of the divided power algebra $H^{\text{even}}(\mathbb{G}_a, \Gamma^*((\mathfrak{gl}_n^{(1)})_{p\alpha})) = H^{\text{even}}(\mathbb{G}_a, k) \otimes \Gamma^*((\mathfrak{gl}_n^{(1)})_{p\alpha})$ and thus the result follows by functoriality of $\Delta_{i,j}$ and of cup product. □

4.5 Other universal classes

If M is a finite dimensional vector space over k and $r \geq 1$, we have a natural homomorphism between symmetric algebras $S^*(M^{\#(r)}) \to S^*(M^{\#(1)})$ induced by the map $M^{\#(r)} \to S^{p^{r-1}}(M^{\#(1)})$ which raises an element to the power p^{r-1}. It is a map of bialgebras. Dually we have the bialgebra map $\pi^{r-1} : \Gamma^{p^{r-1}*}(M(1)) \to \Gamma^*(M(r))$ whose kernel is the ideal generated by $\Gamma^1(M^{(1)})$ through $\Gamma^{p^{r-1}-1}(M^{(1)})$. So π^{r-1} maps $\Gamma^{p^{r-1}a}(M(1))$ onto $\Gamma^a(M(r))$.

Notation 4.6 We now introduce analogues of the classes e_r and $e_r^{(j)}$ of Friedlander and Suslin [11, Theorem 1.2, Remark 1.2.2]. We write $\pi_*^{r-1}(c[ap^{r-1}]) \in H^{2ap^{r-1}}(G, \Gamma^a(\mathfrak{gl}_n^{(r)}))$ as $c_r[a]$. Next we get $c_r[a]^{(j)} \in H^{2ap^{r-1}}(G, \Gamma^a(\mathfrak{gl}_n^{(r+j)}))$ by Frobenius twist.

Lemma 4.7 *The $c_i[a]^{(r-i)}$ enjoy the following properties ($r \geq i \geq 1$)*

1. *There is a homomorphism of algebras $S^*(\mathfrak{gl}_n^{\#(r)}) \to H^{2p^{i-1}*}(G_r, k)$ given on $S^a(\mathfrak{gl}_n^{\#(r)}) = H^0(G_r, S^a(\mathfrak{gl}_n^{\#(r)}))$ by cup product with the restriction of $c_i[a]^{(r-i)}$.*

2. *For $r \geq 1$ the restriction of $c_r[1]$ to $H^{2p^{r-1}}(G_1, \mathfrak{gl}_n^{(r)})$ is nontrivial, so that $c_r[1]$ may serve as the universal class e_r in [11, Thm 1.2].*

Proof 1. By theorem 4.4 we have $(\Delta_{a,b})_*(c_i[a+b]) = (\Delta_{a,b}\pi^{r-1})_*(c[(a+b)p^{r-1}]) = (\pi^{r-1}\otimes\pi^{r-1})_*(\Delta_{ap^{r-1},bp^{r-1}})_*(c[(a+b)p^{r-1}]) = (\pi^{r-1}\otimes\pi^{r-1})_*(c[ap^{r-1}]\cup c[bp^{r-1}]) = c_i[a]\cup c_i[b]$ and thus $(\Delta_{a,b})_*(c_i[a+b]^{(r-i)}) = c_i[a]^{(r-i)}\cup c_i[b]^{(r-i)}$ by pull back along a Frobenius homomorphism. Put $R = H^{\mathrm{even}}(G_r,k)$ and restrict from G to G_r. We view $H^{\mathrm{even}}(G_r,\Gamma^*(\mathfrak{gl}_n^{(r)}))$ as $\Gamma^*_R(R\otimes\mathfrak{gl}_n^{(r)})$. Now the cup product agrees with the usual R-valued pairing between $S^*(\mathfrak{gl}_n^{\#(r)})$ and $\Gamma^*_R(R\otimes\mathfrak{gl}_n^{(r)})$. Thus if $x\in S^a(\mathfrak{gl}_n^{\#(r)})$, $y\in S^b(\mathfrak{gl}_n^{\#(r)})$, then $(xy)\cup(c_i[a+b]^{(r-i)}) = (x\otimes y)\cup((\Delta_{a,b})_*(c_i[a+b]^{(r-i)})) = (x\cup c_i[a]^{(r-i)})(y\cup c_i[b]^{(r-i)})$.
2. In fact if we restrict $c_r[1]$ as in remark 4.1 to $H^{2p^{r-1}}(\mathbb{G}_{a1},(\mathfrak{gl}_n^{(r)})_{p^r\alpha}) = H^{2p^{r-1}}(\mathbb{G}_{a1},k)\otimes(\mathfrak{gl}_n^{(r)})_{p^r\alpha}$, then even that restriction is nontrivial. That is because the Witt vector class generates the polynomial ring $H^{\mathrm{even}}(\mathbb{G}_{a1},k)$, see [15, I 4.26]. □

Corollary 4.8 *Let $n = 2$, $r\geq 1$. Further let A be a commutative k-algebra with SL_n action and J an invariant ideal in A so that the algebra A/J is finitely generated. Then $H^0(SL_n,H^*((SL_n)_r,A/J))$ is a noetherian module over a finitely generated subalgebra of $H^{\mathrm{even}}(SL_n,A)$.*

Proof We may assume A is finitely generated. Put $C = H^0((SL_n)_r,A))$. Then C contains the elements of A raised to the power p^r, so C is also a finitely generated algebra and A/J is a noetherian module over it. By [11, Thm 1.5] $H^*((SL_n)_r,A/J)$ is a noetherian module over the finitely generated algebra

$$R = \bigotimes_{a=1}^{r} S^*((\mathfrak{gl}_n^{(r)})^{\#}(2p^{a-1}))\otimes C.$$

Then, by invariant theory [13, Thm. 16.9], $H^0(SL_n,H^*((SL_n)_r,A/J))$ is a noetherian module over the finitely generated algebra $H^0(SL_n,R)$. By lemma 4.7 we may take the algebra homomorphism $R\to H^*((SL_n)_r,A/J)$ of [11] to be based on cup product with our $c_i[a]^{(r-i)}$. But then the map $H^0(SL_n,R)\to H^*((SL_n)_r,A/J)$ factors, as a linear map, through $H^{\mathrm{even}}(SL_n,A)$. □

4.9 The contraction

Let A be a finitely generated commutative k-algebra on which SL_n acts rationally. Recall that A comes with an increasing filtration $A_{\leq i} = \sum_{\mathrm{ht}(\lambda)\leq i} A_{\leq\lambda}$ whose associated graded is the ring $\mathrm{gr}\,A$. Let $\mathcal{A}$ be the subring of the polynomial ring $A[t]$ generated by the subsets $t^iA_{\leq i}$. This ring $\mathcal{A}$, denoted D by Grosshans, is the coordinate ring of a flat family with general fibre A and special fibre $\mathrm{gr}\,A$ [12, Theorem 13].

4.10 The special fibre

There is a homomorphism of graded algebras $\mathcal{A} \to \operatorname{gr} A$ with kernel $t\mathcal{A}$, mapping $t^i \sum_{\mathrm{ht}(\lambda)=i} A_{\leq\lambda}$ onto $\sum_{\mathrm{ht}(\lambda)=i} A_{\leq\lambda}/A_{<\lambda}$ by 'dropping t^i'. By corollary 4.8 the even part of E_2^{0*} in proposition 3.8 is noetherian over a finitely generated subalgebra R of $H^{\mathrm{even}}(SL_n, \mathcal{A})$. Therefore proposition 3.8 implies in the usual way ([11, Lemma 1.6]) that in fact the abutment $H^*(SL_n, \operatorname{gr} A)$ is noetherian over R.

4.11 The general fibre

One gets a homomorphism $\mathcal{A} \to A$ by substituting a nonzero scalar for t. Let us use the substitution $t \mapsto 1$. On $\mathcal{A}$ we put the filtration with $\mathcal{A}_{\leq m} = \sum_{i\leq m} t^i A_{\leq i}$. Then the associated graded $\operatorname{gr}\mathcal{A}$ is just $\mathcal{A}$ itself. The map $\mathcal{A} \to A$ is compatible with the filtrations, so we get a map of spectral sequences from

$$E(\mathcal{A}): \quad E_1^{ij}(\mathcal{A}) = H^{i+j}(G, \operatorname{gr}_{-i}\mathcal{A}) \Rightarrow H^{i+j}(G, \mathcal{A})$$

to

$$E(A): \quad E_1^{ij}(A) = H^{i+j}(G, \operatorname{gr}_{-i} A) \Rightarrow H^{i+j}(G, A).$$

Note that by the construction of the Grosshans filtration $H^0(G, \operatorname{gr}_{-i} A)$ vanishes for $i \neq 0$. Further $E_1^{**}(A)$ is finitely generated by theorem 1.1 and therefore there are for each m only finitely many non-zero $E_1^{m-i,i}(A)$. This makes that, even though the spectral sequence is a second quadrant spectral sequence, the abutment will be finitely generated as soon as $E_\infty^{**}(A)$ is.

Theorem 4.12 (Cohomological invariant theory in rank one) *Let A be a finitely generated commutative k-algebra on which SL_2 acts rationally by algebra automorphisms. Then $H^*(SL_2, A)$ is finitely generated as a k-algebra.*

Proof We combine the above. The spectral sequence

$$E(\mathcal{A}): \quad E_1^{ij}(\mathcal{A}) = H^{i+j}(G, \operatorname{gr}_{-i}\mathcal{A}) \Rightarrow H^{i+j}(G, \mathcal{A})$$

is pleasantly boring: It does not just degenerate, even its abutment is the same as its E_1. The spectral sequence $E(A)$ is a module over it [18]. In particular, $E(A)$ is a module over the finitely generated subring R of $H^{\mathrm{even}}(SL_n, \mathcal{A})$ over which $E_1^{**}(A)$ is noetherian by 4.10. (Yes, the homomorphism $R \to E_1^{**}(A)$ agrees with the homomorphism $E_1^{**}(\mathcal{A}) \to E_1^{**}(A)$.) So the usual argument (see proof of [11, Lemma 1.6] or [9, Lemma 7.4.4]) shows that $E(A)$ stops and that $E_\infty^{**}(A)$ is noetherian over R. $\square$

Corollary 4.13 *Let A be a finitely generated commutative k-algebra on which SL_2 acts rationally by algebra automorphisms. Assume that A has a good filtration. Let M be a finitely generated A module with a compatible SL_2 action. Then M has finite good filtration dimension.*

Proof Write $G = SL_2$. If we tensor A with the multicone $k[G/U]$ of example 2.4, then the result is a finitely generated G-acyclic k-algebra $A \otimes k[G/U]$ over which we have a finitely generated module $M \otimes k[G/U]$. As $H^*(G, S^*_{A\otimes k[G/U]}(M \otimes k[G/U]))$ is finitely generated by the theorem, $H^i(G, M \otimes k[G/U])$ vanishes for $i >> 0$. But this means that M has finite good filtration dimension. $\square$

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